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THE QUATERNION METHOD OF REGULARIZING INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY†

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Using quaternions [1], a new method of regularizing integral equations of the theory of elasticity is proposed which is identical for plane and spatial problems. The regularizors and the regularized equations are presented. The method does not use the theory of a symbol, as was done earlier [2–4], or the transition to the complex form [5]. The quaternion technique has not been applied so far to integral equations of the theory of elasticity but has been used to obtain the general solutions of Lamé's equations.‡

Definition 1. Quaternions are numbers of the form $a_0+a_ie_i=a_0+a$, where e_i are the same square root of minus one, a_0 and a_i are real numbers ($i=1, 2, 3$), and a is an imaginary quaternion.

Operations with quaternions are defined by means of operations with the square root of minus one. If the square root of minus one is interpreted as unit vectors of a Cartesian basis, the operation of multiplication can be expressed in terms of scalar and vector products $e_i^2=-1$, $e_{ijk}e_k=e_ie_j$ $(i\neq j)$, where e_{ijk} is the Levi–Civita symbol. The table of multiplication enables the product of arbitrary quaternions $z_1=a_0+a$ and $z_2=b_0+b$ to be interpreted in terms of the operations of the scalar and vector products $z_1z_1=a_0b_0+a_0b+b_0a+a\times b-a\cdot b$. The product of quaternions is non-commutative and the associative rule holds.

Let $z(x_1, x_2, x_3)$ be any quaternion function and $\nabla=e_i\partial/\partial x_i$ be the quaternion Hamilton operator.

Definition 2. The function z will be called a quaternion analytic (K-analytic) function if it satisfies the relation $\nabla z=0$.

In the plane case, a function of two variables $z(x_1, x_2)=z_0(x_1, x_2)+e_iz_i(x_1, x_2)$ but having three square roots of minus one will be considered.

Let l be a closed piecewise smooth curve in the S plane, S^+ the interior region bounded by the curve l, S^- the exterior region, τ and n vectors which are, respectively, tangent and normal to the curve l, and let k be a vector which is perpendicular to the S plane, $k=n\times\tau$.

Vectors τ, n and k may be resolved with respect to the basis e_1, e_2, e_3 where the unit vectors e_1 and e_2 belong to S, $e_3\|k$. The components may be interpreted as imaginary quaternion functions and a number of theorems may be formulated.

Theorem 1. Let z and q be arbitrary quaternion functions of two variables x_1, x_2 differentiable in S^+ and let n be an imaginary quaternion. The equality

$$\int_l znq\,dl=\int_{S^+} z\nabla q\,dS+\int_{S^+}\overline{\bar{q}\,\nabla\,\bar{z}}\,dS \tag{1}$$

then holds [an over bar indicates the operation of conjugation (the sign before each of the imaginary unities of a quaternion is replaced by the opposite one)]. The theorem is proved by using Stokes's formula which connects surface and curvilinear integrals.

We put $h=\nabla_x\ln|r|$, where $r=|x-y|$, $x, y\in S$.

Theorem 2. Let $q(x)$ $(x\in S^+)$ be an arbitrary quaternion which is continuous up to the boundary and has derivatives bounded in S^+. We then have

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‡GRIGOR'YEV Yu. M., Solving spatial statics problems of the theory of elasticity by the methods of the theory of quaternion functions. Candidate disseration, Novosibirsk, 1985.

$$\int_l hn_x q_x dl_x - \int_{S^+} h\nabla_x q(x)\, dS_x = \begin{cases} 0, & y \in S^- \\ 2\pi q(y), & y \in S^+ \end{cases} \tag{2}$$

This is proved by substituting the quaternion $z = h$ into Eq. (1). If the point $y \in S^+$, it is necessary to remove a circle of radius ϵ, centred at y, from the S^+ domain, to write Eq. (1) and to let ϵ go to zero. Note that $\nabla_x h = 0$ everywhere except at the point $x = y$.

Theorem 3. Let $q(x)$ be an arbitrary quaternion which is continuous up to the boundary and has derivatives bounded in S^-. Let $\lim q(x) = 0$ as $|x| \to \infty$. This inequality, which differs from (2) in the sign before the first integral and in the notation of the S^+ and S^- domains then holds.

Theorem 4. (The analogue of Gauss' integral in the quaternion field.) For a piecewise-smooth curve l the equality

$$J(y) = \int_l hn_x dl_x = \begin{cases} 0, & y \in S^- \\ -\omega, & y \in l \\ -2\pi, & y \in S^+ \end{cases}$$

holds, where ω is the angle enclosed by the tangents to the curve at the point $y \in l$.

Definition 3. The integral

$$Q(y) = \int_l hn_x q(x)\, dl_x, \quad y \in S^{\pm}$$

is called the analogue of the potential of a double layer (or the analogue of a Cauchy-type integral).

Here $q(x)$ is an arbitrary quaternion. For $y \in l$ the integral $Q(y)$ is singular. Let us consider the operator A

$$Aq = \pi^{-1} Q(y), \quad y \in l' \tag{3}$$

Theorem 5. (The analogue of the Cauchy integral formula.) Let $q(x)$ be a K-analytic function in the S^+ domain; then its representation by means of the boundary value

$$Q(y) = \begin{cases} 0, & y \in S^- \\ -\omega q(y), & y \in l \\ -2\pi q(y), & y \in S^+ \end{cases}$$

holds.

This is proved from Theorems 2 and 4.

From Theorems 2 and 5 a similar result is obtained for functions which are K-analytical in the S^- domain.

Hence it holds that for the boundary values of such functions given in the S^+ and S^- domains we have, respectively,

$$Aq = -\pi^{-1}\omega q, \qquad Aq = \pi^{-1}(2\pi - \omega) q \tag{4}$$

These equalities may be combined into one

$$Aq = \pm q \tag{5}$$

for the Lyapunov curve $\omega = \pi$.

Theorem 6. The function $Q(y)$ is K-analytic in both the S^+ and S^- domains.

To obtain the proof it is necessary to apply to operator ∇_y to the function $Q(y)$, substitute it into the integrand and take into account that $\nabla_y h = -\nabla_x h$ and $\nabla_x h = 0$ for $x \neq y$.

Theorem 7. Let a quaternion $q(x)$ be specified on a piecewise-smooth curve l which may be continued into the S^+ and S^- domains retaining the Hölder condition†

†The conditions of Theorem 7 may not be satisfied uniquely. If a function $q(x)$ satisfies the Hölder condition on the boundary l, then it has even a harmonic continuation into the S^+ and S^- domains.

$$|q(x') - q(x'')| \leq c\,|x' - x''|^{\alpha}$$
$$c > 0, \quad 0 < \alpha < 1, \quad x',\ x'' \in S^{+} \cap S^{-}$$

Then for boundary values of the function $Q(y)$ we have

$$Q^{+} = (-2\pi + \omega)\,q + \pi A q, \quad Q^{-} = \omega q + \pi A q \tag{6}$$

where $Q^{\pm}$ are the limit values as the point y approaches the boundary from the $S^{\pm}$ domains, respectively.

The theorems given above resemble outwardly the set of well-known theorems of the theory of analytic functions, namely those of potential theory. The distinction lies in the presence of the quaternion functions and the quaternion products in them.

Theorem 8. Let $q(x)$ be an arbitrary quaternion which satisfies the conditions of Theorem 7. We then have

$$A^{2} q = \pi^{-2}(2\omega\pi - \omega^{2})\,q + \pi^{-1}(2\pi - 2\omega)\,Aq \tag{7}$$

For the Lyapunov curve we obtain

$$A^{2} q = q \tag{8}$$

The proof of identity (7) follows from the first or the second of Eqs (4) after substituting the corresponding value Q^{+} or Q^{-} [given in (6)] for q. It is possible to make this substitution because $Q^{\pm}$ are the limit values of the K-analytic function Q.

Theorem 9. Equality $A = A^{-1}$ holds.

The proof follows from (8) and holds for the Lyapunov curves l.

Putting $Aq = p$ we obtain a pair of transformations from (8)

$$Aq = p, \quad Ap = q \tag{9}$$

Let p_0, p', q_0 and q' be real and imaginary parts of the quaternion p and q. Using the vector and scalar interpretations of the multiplication of quaternions and separating the imaginary and real parts, it is possible to rewrite Eqs (9) in vector form.

We will introduce the operators

$$Bq_0 = \pi^{-1}\int_l q_0 h \cdot n_x\,dl_x, \qquad Cq_0 = \pi^{-1}\int_l q_0 h \times n_x\,dl_x$$
$$Dq' = \pi^{-1}\int_l [-q'h \cdot n_x + (h \times n_x) \times q']\,dl_x$$
$$Fq' = \pi^{-1}\int_l q \cdot (h \times n)\,dl_x \tag{10}$$

Then Eqs (9) take the form

$$-Bq_0 - Fq' = p_0, \quad Cq_0 + Dq' = p' \tag{11}$$
$$-Bp_0 - Fp' = q_0, \quad Cp_0 + Dp' = q' \tag{12}$$

Henceforth we will omit the primes on the vectors p' and q'.

Theorem 10. The operators C, D and F are singular, and the operator B is completely continuous.

The integral Bp_0 is known as the double-layer potential.

Up to now, it has been implied that $q = q(x_1, x_2)$ $(x_1, x_2 \in S)$ and that the vector q has any direction. Let us now assume that it lies in the S plane.

Theorem 11. Let $q \perp k$. Then we have

$$Fq = 0, \quad Cq_0 \parallel k, \quad Dq \perp k \tag{13}$$
$$FDq = 0, \quad q_0 + B^{2} q_0 = -FCq_0, \quad DCq_0 = CBq_0 \tag{14}$$
$$D^{2} q = q \tag{15}$$

Proof. For $x, y \in S$ the vector h lies in the S plane. Hence Eq. (13) follows at once from definition (10) of the operators F, C and D.

We find $p_0 = -Bq_0$ and $p = Cq_0 + Dq$ from Eqs (11) and (13). Substituting p_0 and p into (12) and using the fact that q_0 and q are arbitrary, we obtain Eqs (14) and (15).

The integral equations of the plane state of strain, obtained by using the Somigliana identity (the direct statement), have the form

$$0{,}5u + Gu = \tfrac{1}{2}u - \alpha Du - \beta Tu = Kf \tag{16}$$

where $\alpha = (1-\nu)/(4(1-\nu))$, $\beta = 1/(2(1-\nu))$, and $u(x)$ and $f(x)$ are the displacement and stress vectors defined at the points x of the boundary l.

The operator D is given by formula (10), and

$$Kf = \int_l f(x) \cdot U(x,y)\, dl$$

$$U(x,y) = -\frac{1+\nu}{4\pi E(1-\nu)}\left[(3-4\nu)I \ln|r| - rh\right]$$

where E is Young's modulus, ν is Poisson's ratio, I is the unit matrix, and rh is the dyadic product of the vectors r and h [6].

Operator T in (16) is completely continuous, the form of which is henceforth not important.

All the singularity of integral equation (16) is contained in the operator D.

Theorem 12. The operator $R = 0.5I + \alpha D$ is the equivalent regularizor of Eq. (16). The regularized equation has the form

$$(0{,}25 - \alpha^2)u - \beta RTu = RKf \tag{17}$$

Proof. Multiplying (16) by the operator R and taking into account identity (15), we obtain Eq. (17). The complete continuity of the operator $R \cdot T$ follows from the fact that one of the factors, that is the operator T, is completely continuous.

By virtue of (15), which holds for every q, the homogeneous equation $R_\varphi = 0$ has only a trivial solution for $\alpha \neq \pm 0.5$. For this reason [3] the regularization will be equivalent.

Theorem 13. The operator $R' = 0.5I - G$ is the equivalent regularizor of Eq. (16). The regularized equation has the form

$$0{,}25u - G^2u = R'Kf \tag{18}$$

Proof. Since we can always add any completely continuous operator to the regularizor [2], the operator $R' = R + \beta T$ is a regularizor. Squaring the operator G and taking into consideration identity (15), we can write the regularized equation (18) explicitly as

$$G^2 = (\alpha D + \beta T)^2 = \alpha^2 I + \beta^2 T^2 + \alpha\beta(DT + TD) \tag{19}$$

The regularized equation (18) may be preferable for calculations because

(i) the norm of the operator G^2 is smaller than the norm of the operator G, and

(ii) the number of arithmetic operations decreases when Eq. (19) is used (since we skip the calculation of D^2).

For the indirect statement, the first and the second problems of the theory of elasticity are reduced to the problem of finding the density vectors φ and ψ of certain potentials

$$\pm 0{,}5\varphi + G\varphi = u, \qquad \pm 0{,}5\psi + G^*\psi = f \tag{20}$$

The operator G^* is conjugate with respect to the operator G. It can be shown that $G^* = +\alpha D + T_1$, where T_1 is a specified completely continuous operator. Therefore the scheme for regularizing Eqs (20) is identical with that for regularizing Eq. (16). In particular, any of the operators $\pm 0.5I + \alpha D$, $\pm 0.5I - G$ and $\pm 0.5I - G^*$ is the regularizor of Eq. (20). All these operators are identical up to the completely continuous summands.

The quaternion technique of regularization may also be extended to the integral equations of spatial problems of the theory of elasticity. Assertions are known concerning the need to distinguish the cases of one and several variables in the theory of regularization [2, p. 7; 3, p. 197].

For the case of spatial problems Theorems 1–10 also hold if a number of changes are carried out.

Instead of the plane domains S^+ and S^- bounded by a contour l, we will consider the three-dimensional domains V^+ and V^- enclosed by a surface S. The quantity $-1/|r|$ is substituted for the function $\ln|r|$ and 2π is substituted for π. These substitutions have also to be carried out in the operators B, C, D and F.

For the case of spatial problems, Theorems 1–10 may partly be obtained by formulating the results of [7, 8] in quaternion form.

The integral equations of spatial problems of the theory of elasticity presented in operator form are (16) and (20). The operators G, G^*, K may be taken from [6].

It is important that the representations $G = -\alpha D + T_1$ and $G^* = -\alpha D + T_2$ where T_1 and T_2 are the specified completely continuous operators, are made, i.e. equalities $D = -G/\alpha = -G^*/\alpha$ hold, the last of them being known [9].

Instead of Theorem 11 we prove the following one.

Theorem 14. For the operators B, C, D and F the identities

$$\begin{aligned} B^2q_0 - FCq_0 &= q_0, \quad -BFq + FDq = 0 \\ -CBq_0 + DCq_0 &= 0, \quad -CFq + D^2q = q \end{aligned} \tag{21}$$

hold.

Proof. Eliminating the quantity p_0 and p' from Eqs (11) and (12) and then substituting zero values for q_0 and q' successively, we obtain the identities (21).

Theorem 15. The operator $D^3 - D$ is completely continuous.

Proof. The following corollary may be obtained from Eqs (21)

$$D^3q - Dq = CBFq \tag{22}$$

The operator $D^3 - D$ is completely continuous because it may be replaced by the composition CBF which contains the completely continuous multiplier B.

Theorem 16. The operator

$$\tfrac{1}{4}\alpha^{-2}(1 - 4\alpha^2)I + \tfrac{1}{2}\alpha^{-1}D + D^2 = R \tag{23}$$

is the regularizor of Eq. (16).

The regularized equation has the form

$$0{,}125\alpha^{-2}(1 - 4\alpha^2)u - \alpha CBFu + RT_1u = RKf \tag{24}$$

Proof. Let R be a second degree polynomial in D with undetermined coefficients. Multiplying Eq. (16) by R and using (22) we select the coefficients so as to eliminate the powers of the operator D. We obtain the statements of the theorem.

Substituting $-G/\alpha$ or G^*/α for D in (23) we obtain the regularizors in terms of the original operators G or G^*, for example

$$R' = \tfrac{1}{2}(1 - 4\alpha^2)I - \tfrac{1}{2}G + G^2 \tag{25}$$

As the spectrum of G is real, the equation $R'_\varphi = 0$ has only a trivial solution. Hence R' is the equivalent regularizor.

For Eqs (20) the regularizors may be obtained from (25). To do this we multiply Eqs (20) by $\delta = \pm 1$. This gives

$$\varphi + \delta G\varphi = \delta u, \qquad \psi + \delta G^*\psi = \delta u$$

Substituting δG or δG^* for G in (25) we obtain the equivalent regularizors which are suitable for any of Eqs (20).

Some of the regularizors of the type (25) were obtained previously in [4] by using the theory of a symbol.

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